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A Finite-Difference/Galerkin Finite-Element Solution of a Turbulent Boundary Layer

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Nomenclature

c_f = local skin-friction coefficient $2\tau_w/\rho u_e^2$
f = dimensionless stream function
(M^e) = element characteristic matrix
(M) = system characteristic matrix
$\{m^e\}$ = element characteristic vector
$\{m\}$ = system characteristic vector
p = static pressure
R_x = Reynolds number xu_e/ν
u,v = mean velocity components in x and y directions
x,y = coordinates measured along and normal to the body
β = pressure gradient parameter $(2\xi/u_e)$ $(du_e/d\xi)$
δ = boundary-layer thickness
ϵ = eddy viscosity
ϵ^+ = dimensionless eddy viscosity ϵ/ν
μ = dynamic viscosity
ν = kinematic viscosity
ξ,η = transformed x,y coordinates
ρ = fluid mass density
$-\rho\overline{uv}$ = Reynolds shear-stress term
τ = shear stress
ψ = streamfunction
ζ = $(2\xi)^{1/2}/\mu$

Subscripts

e = outer edge of boundary layer
i = inner region
o = outer region
w = wall

Superscript

$'$ = derivatives with respect to η

I. Introduction

In recent years, with the advent of high-speed computers, much progress in the numerical solutions of the turbulent boundary-layer equations, in their differential form, has been made. Finite-difference schemes have been used extensively,[1] and sufficiently accurate results were obtained for two-dimensional incompressible turbulent flows,[2-4] compressible turbulent flows,[5-8] and turbulent flows with heat and mass transfer.[9,10] Boussinesq's eddy-viscosity concept and Prandtl's mixing length theory have been used widely in these methods in order to relate the Reynolds shear-stress terms to the mean velocities.

In Ref. 11, a finite-difference/Galerkin finite-element method was presented for the solution of compressible laminar flows. In the present Note, the application of the method is extended to the solution of incompressible turbulent flows. The governing equations are presented and transformed first; the application of the finite-difference method in the streamwise direction and the Galerkin finite-element method through the boundary layer then are introduced; and the method of solution is discussed. Numerical results are given for turbulent flows over a flat plate and are compared with the finite-difference solutions of other investigators.

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II. Theoretical Formulation

A. Governing Equations

Neglecting the normal stress terms, the boundary-layer equations for steady, incompressible, turbulent two-dimensional flows can be written as follows[1-2]:

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}+\rho\overline{uv}\right) \tag{2}$$

The boundary conditions are as follows.[12] For no slip on the wall,

$$u(x,0)=0 \tag{3a}$$

For no mass transfer on the wall or for flows with mass transfer on the wall, respectively,

$$v(x,0)=0; \quad v(x,0)=v_w(x) \tag{3b}$$

For smooth merging of the velocity into the freestream value,

$$\lim_{y\to\infty} u(x,y)=u_e(x) \tag{3c}$$

In the following, Boussinesq's eddy-viscosity concept is used in order to relate the Reynolds shear stress to the mean velocity, i.e.,

$$-\rho\overline{uv}=\rho\epsilon\frac{\partial u}{\partial y} \tag{4}$$

and, according to this concept, the turbulent boundary layer will be regarded as composed of two regions with separate expressions for ϵ in each region. In the inner region, an expression for ϵ based on Prandtl's mixing length theory, and incorporating the modifications introduced by Van Driest[13] to account for the viscous sublayer close to the wall and by Cebeci[14] to account for flows with pressure gradients, is used and reads

$$\epsilon_i=(0.4y)^2\left[1-\exp\left\{-y\left(\frac{\tau_w}{\rho}+\frac{dp}{dx}\frac{y}{\rho}\right)\Big/26\nu\right\}\right]^2\left|\frac{\partial u}{\partial y}\right| \tag{5}$$

In the outer region, a constant eddy-viscosity expression with Klebanoff's intermittency factor[15] is used:

$$\epsilon_0 = 0.0168 \left| \int_0^\infty (u_e - u)\,dy \right| \left[1 + 5.5\left(\frac{y}{\delta}\right)^6 \right]^{-1} \quad (6)$$

The inner and outer eddy-viscosity expressions are matched by the requirement of continuity. Now, applying the Levy-Lees transformations

$$d\xi = \rho\mu u_e dx; \quad d\eta = [\rho u_e / (2\xi)^{1/2}]dy \quad (7)$$

in order to remove the singularity at $x=0$ and to stretch the coordinates in the x and y directions, and defining a dimensionless streamfunction

$$f(\xi,\eta) = \psi(x,y)/(2\xi)^{1/2} \quad (8)$$

the boundary-layer equations reduce to

$$[(1+\epsilon^+)f'']' + ff'' + \beta[1-(f')^2] = 2\xi\left[f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right] \quad (9)$$

We notice that the introduction of the streamfunction eliminates the continuity equation, since it is satisfied automatically. In terms of the transformed coordinates, the boundary conditions (3) are

$$f'(\xi,0) = 0 \quad (10a)$$

$$f(\xi,0) = 0 \text{ or } f(\xi,0) = -\left(\frac{1}{2\xi}\right)^{1/2}\int_0^\xi \left[\frac{v_w}{\mu u_e}\right]d\xi \quad (10b)$$

$$\lim_{\eta\to\infty} f'(\xi,\eta) = 1 \quad (10c)$$

and the eddy-viscosity expressions (5) and (6) read

$$\epsilon_i^+ = 0.16\zeta|f''|\eta^2\{1-\exp[-(\zeta\eta/26)(|(f''_w/\zeta)-(\beta\eta/\zeta)|)^{1/2}]\}^2 \quad (11)$$

$$\epsilon_o^+ = 0.0168\left|\int_0^\infty (1-f')\,d\eta\right|\left\{1+5.5\left(\frac{\eta}{\eta_\infty}\right)^6\right\}^{-1} \quad (12)$$

B. Conversion to the Ordinary Differential Equation

Because of the parabolic nature of the boundary-layer equations,[12] the solution of the problem will be obtained by marching up in the streamwise direction. Thus, dividing the boundary-layer field into various stations along the ξ direction, and replacing the streamwise derivatives of the momentum equation (9) by their equivalent two-point finite-difference approximation, we get, for the station ξ_n, the following nonlinear ordinary differential equation:

$$[(1+\epsilon^+)f'']' + ff'' + \beta[1-(f')^2] = \alpha[f'(f'-f'_{n-1}) - f''(f-f_{n-1})] \quad (13)$$

where $\alpha = (\xi+\xi_{n-1})/(\xi-\xi_{n-1})$, and for simplicity the subscript n has been dropped. We notice that the quantities with subscript $(n-1)$ in Eq. (13) are assumed to be known values from the solution at $\xi = \xi_{n-1}$.

C. Linearization of the Momentum Equation

Linearization of the momentum equation will be made through the application of Newton's iterative method. Thus, writing, at the iteration $(i+1)$ for the field variable f and its derivatives, expressions in the form

$$f^{(i+1)} = f^{(i)} + \delta f^{(i)} \quad (14)$$

and inserting such expressions in the momentum equation (13), the following ordinary linear equation is obtained:

$$q_1\delta f''' + q_2\delta f'' + q_3\delta f' + q_4\delta f + q_5 = 0 \quad (15)$$

where

$$q_1 = (1+\epsilon^+)$$

$$q_2 = (1+\epsilon^+)' - \alpha_n f_{n-1} + (1+\alpha_n)f$$

$$q_3 = \alpha_n f'_{n-1} - 2(\beta+\alpha_n)f'$$

$$q_4 = (1+\alpha_n)f''$$

$$q_5 = (1+\epsilon^+)f''' + [(1+\epsilon^+)' - \alpha_n f_{n-1}]f'' + (\alpha_n f'_{n-1})f' + (1+\alpha_n)ff'' - (\beta+\alpha_n)(f')^2 + \beta \quad (16)$$

The subscript i has been dropped from Equation (15) and relations (16). Furthermore, if the initial approximation $f^{(o)}$ satisfies boundary conditions (10), then the boundary conditions on (15) will be

$$\delta f(\xi,0) = 0; \; \delta f'(\xi,0) = 0; \; \delta f'(\xi,\eta_\infty) = 0 \quad (17)$$

D. Galerkin Finite Element

At station $\xi = \xi_n$ and for the iteration $(i+1)$, the Galerkin finite-element method[16-18] will be used in order to solve the linearized equation (15). One-dimensional elements are used, and we write, for the field variable δf, within each element, the following approximation:

$$\delta f = \sum_{i=1}^{2}\sum_{j=0}^{2} H_{ji}^{(2)}\delta f_i^{(j)} \quad (18)$$

where $H_{ji}^{(2)}$ are second-order Hermitian polynomials and are given by

$$H_{01}^{(2)} = (1-10z^3+15z^4-6z^5) \quad (19a)$$

$$H_{11}^{(2)} = L(z-6z^3+8z^4-3z^5) \quad (19b)$$

$$H_{21}^{(2)} = L^2(0.5z^2-1.5z^3+1.5z^4-0.5z^5) \quad (19c)$$

$$H_{02}^{(2)} = (10z^3-15z^4+6z^5) \quad (19d)$$

$$H_{12}^{(2)} = L(-4z^3+7z^4-3z^5) \quad (19e)$$

$$H_{22}^{(2)} = L^2(0.5z^3-z^4+0.5z^5) \quad (19f)$$

and $L = (\eta_2-\eta_1)$, $z = (\eta-\eta_1)/(\eta_2-\eta_1)$, $\delta f_i^{(0)} = \delta f_i$, $\delta f_i^{(1)} = \delta f'_i$, $\delta f_i^{(2)} = \delta f''_i$, $i=1,2$. The subscripts 1 and 2 stand for the element nodal points. We notice that the interpolation functions used here preserve C^2 continuity for Eq. (15), thus satisfying the compatibility and completeness requirements of the finite-element method.[18] Substituting Eq. (18) into Eq. (15) and applying the Galerkin finite-element method, we obtain for each element the following matrix equation:

$$(M^e)\{\delta F^e\} = \{m^e\} \quad (20)$$

where

$$\{\delta F^e\} = (\delta f_1 \delta f'_1 \delta f''_1 \delta f_2 \delta f'_2 \delta f''_2)^T \quad (21)$$

The element characteristic matrix (M^e) and the element characteristic vector $\{m^e\}$ are calculated using Gaussian numerical integrations. Now, using the standard assembly technique of the finite-element method and applying the appropriate boundary conditions, we obtain the final systems

Table 1 Local skin-friction coefficient $c_f \times 10^2$ for turbulent flow over a flat plate

	Present method				Keller-Cebeci box method[2]	Cebeci-Smith[20]
	η points					
$R_x \times 10^{-6}$	20	23	27	31	31	230
0.51	0.3894	0.3969	0.4014	0.4043	0.40648	0.39954
0.68	0.3674	0.3781	0.3824	0.3844	0.38708	0.38002
0.86	0.3497	0.3640	0.3683	0.3700	0.37347	0.36614
1.03	0.3362	0.3539	0.3583	0.3599	0.36205	0.35536
1.28	0.3202	0.3426	0.3473	0.3488	0.34850	0.34244
1.50	0.3086	0.3346	0.3397	0.3410	0.34028	0.33429
1.71	0.2988	0.3282	0.3336	0.3348	0.33368	0.32741
1.88	0.2917	0.3237	0.3293	0.3304	0.32873	0.32272

equations as

$$(M)\{\delta F\} = \{m\} \tag{22}$$

E. Method of Solution

At $\xi = 0$, the flow is assumed to be laminar, i.e., $\epsilon^+ = 0$, and we assume as an initial guess for the solution a linear variation of the velocity through the boundary layer. An iterative process then is started using Eqs. (22) and (14) and is stopped based on a convergence criterion of the shear stress at the wall. The following criterion was adopted for convergence: $|f_w''^{(I)} - f_w''^{(i+I)}| / |f_w''^{(i)}| < 0.0001$.

At any other location $\xi = \xi_n$, the initial values of the solution are taken from the final solution at $\xi = \xi_{n-1}$. When the flow becomes turbulent, at any specified station, the eddy-viscosity expressions are activated. But, since these expressions contain terms that depend on the field solution, a further iteration process is necessary. For the first iteration, the eddy-viscosity expressions are calculated from the final solution of the previous station, and the inner and outer regions are established from the continuity requirements. The momentum equation then is solved, and more accurate values for the eddy-viscosity expressions are found. The whole process is then repeated until convergence is attained based on the same previously mentioned criterion. In the numerical calculation performed, the number of iterations was of the order of 3.

III. Numerical Results

Results of some computations using the present method are reported in this section in order to show the applicability of the method, its accuracy, and convergence. For simplicity, only turbulent flow over a flat plate is considered. At the leading edge, the flow is assumed to be laminar and then becomes turbulent at the next station. The calculations were performed for a variable grid spacing along the ξ direction, corresponding to Reynolds numbers $R_x \times 10^{-6} = 0$, 0.1, 0.17, 0.26, 0.34, 0.51, 0.68, 0.86, 1.03, 1.28, 1.5, 1.71, and 1.88. Such spacing has been chosen in order to compare the results here obtained with those of other investigators,[2,19,20] where calculations were performed for the same ξ locations. The results obtained using the present formulation are given in Table 1 and are compared with the Keller-Cebeci[2] finite-difference box method, which is claimed to be one of the most efficient finite-difference solutions of turbulent boundary-layer flows,[2,19,21] and the Cebeci-Smith finite-difference method.[20] The results here obtained showed that the convergence of the present method is quite rapid, and the results are as accurate as the finite-difference methods.

IV. Conclusions

A finite-difference/Galerkin finite-element method for the solution of the turbulent incompressible boundary-layer problem has been presented. Boussinesq's eddy-viscosity concept has been used in order to relate the Reynolds shear stress to the mean velocity. The method has been applied for the calculation of turbulent flows over a flat plate. The numerical results obtained showed the validity and the effectiveness of the proposed formulation.

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Integral Equation Formulation for Transonic Lifting Profiles

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Introduction

THE integral equation method, originated by Oswatitsch[1] for studying the direct problem of steady inviscid irrotational transonic flow past a thin symmetric profile at zero incidence, is now well established. It has been extended and modified by a number of authors, such as Gullstrand et at.,[2] Zierep,[3] Nϕrstrud,[4] Niyogi,[5], Nixon and Hancock,[6] Frohn,[7] and others. A controversy originated recently, from a criticism by Nixon[8] regarding the correctness of the integral equation formulation of Nϕrstrud[9] for the lifting profile flow problem. According to Nixon, Nϕrstrud's formulation leads to a nonunique solution. Nϕrstrud has given two, apparently different, integral equation formulations,[4,9] for the transonic lifting profile flow problem. The purpose of the present Note is to show that the formulations of Nixon and Hancock and Nϕrstrud[4] are equivalent.

Different Forms of Formulation

The problem under consideration is to study the steady inviscid irrotational transonic flow of a compressible fluid past a thin unsymmetric profile at small incidence, with freestream Mach number $M_\infty < 1$. The transonic small disturbance continuity equation and the irrotationality condition for this case may be reformulated in terms of a system of two-dimensional nonlinear singular integral equations. Three different forms of integral equation formulations for this problem have been given in Refs. 4, 9, and 6, among which the present Note shows that the formulations in Refs. 4 and 6 are equivalent.

According to Nϕrstrud,[4] the reduced perturbation potential $\Phi(X,Y)$ of the above problem, satisfies the following system of four equations

$$\Phi_X^+(X,0) = \bar\Phi_X^+(X,0) - \frac{1}{2\pi}\int\int_{-\infty}^{\infty} [\Phi_\xi^+\Phi_{\xi\xi}^+ + \Phi_\xi^-\Phi_{\xi\xi}^-]\frac{\partial}{\partial\xi}\left(\ln\frac{1}{R}\right)\mathrm{d}\xi\,\mathrm{d}\eta \tag{1a}$$

$$\Phi_X^-(X,0) = \bar\Phi_X^-(X,0) \tag{1b}$$

$$\Phi_Y^-(X,0) = \bar\Phi_Y^-(X,0) - \frac{1}{2\pi}\int\int_{-\infty}^{\infty} [\Phi_\xi^+\Phi_{\xi\xi}^- + \Phi_\xi^-\Phi_{\xi\xi}^+]\frac{\partial}{\partial\eta}\left(\ln\frac{1}{R}\right)\mathrm{d}\xi\,\mathrm{d}\eta \tag{1c}$$

$$\Phi_Y^+(X,0) = \bar\Phi_Y^+(X,0) \tag{1d}$$

where the superscripts + and − denote, respectively, the symmetric and antisymmetric parts defined by

$$\Phi = \Phi^+ + \Phi^-, \qquad \Phi^+(X,Y) = \Phi^+(X,-Y), \qquad \Phi^-(X,Y) = -\Phi^-(X,-Y) \tag{2}$$

and the subscripts X and Y denote partial differentiation with respect to X and Y, respectively, and

$$R = \sqrt{(X-\xi)^2 + (Y-\eta)^2}$$

The overbar on Φ, denotes a solution of the Laplace equation. It is an unknown of the problem to be determined along with the unknown nonlinear solution Φ by means of the boundary conditions and the irrotationality condition. Furthermore, $\bar\Phi$ should not be confused with the Prandtl solution Φ_P, which is a known quantity, being the solution in reduced coordinates of the Laplace equation and the same boundary condition as that of the nonlinear problem. The reduced velocity potential Φ is related to the true velocity potential φ by

$$\Phi(X,Y) = K(\varphi - u_\infty x - v_\infty y)/[(1-M_\infty^2)u_\infty] \tag{3a}$$

and the reduced coordinates denoted by the corresponding capital letters by

$$X = x, \qquad Y = y\sqrt{1-M_\infty^2} \tag{3b}$$

and u_∞, v_∞ denote freestream velocity components.

The parameter K is a function of the freestream Mach number M_∞, for which different approximate values may be used. For example, Oswatitsch uses the value

$$K = (1-M_\infty^2)/[(1/M_\infty^*) - 1] \tag{4}$$

M_∞^* denoting the critical freestream Mach number, and Spreiter takes

$$K = M_\infty^2(\gamma + 1) \tag{5}$$

Other useful values of K are given in Ref. 4. In the above formulation, there are eight unknowns, viz., Φ_X^+, Φ_X^-, Φ_Y^+, Φ_Y^- and $\bar\Phi_X^+$, $\bar\Phi_X^-$, $\bar\Phi_Y^+$, $\bar\Phi_Y^-$.

A second integral formulation for the above problem has been given by Nixon and Hancock[6] who obtained the following equations for shock-free flow

$$U(X,+0) + U(X,-0) - \frac{U^2(X,+0) + U^2(X,-0)}{4} = \frac{1}{\pi}\int_0^1 \frac{\Delta V(\xi)}{X-\xi}\mathrm{d}\xi - \lim_{y\to+0}\frac{1}{4\pi}\int_S\int \psi_{\xi X}(X,\xi;Y,\eta)[U^2(\xi,\eta) + U^2(\xi,-\eta)]\mathrm{d}S \tag{6a}$$

and

$$V(X,+0) + V(X,-0) = -\frac{1}{\pi}\int_0^1 \frac{U(\xi,+0) - U(\xi,-0)}{X-\xi}\mathrm{d}\xi - \lim_{Y\to+0}\frac{1}{4\pi}\int_S\int \psi_{\xi Y}(X,\xi;Y,\eta)[U^2(\xi,\eta) - U^2(\xi,-\eta)]\mathrm{d}S \tag{6b}$$

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